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Shifted Chebyshev Spectral Collocation Method for Solving One-Dimensional Advection-Diffusion Equation

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Abstract

In this paper, a numerical method for the solution of one-dimensional advection-diffusion equation is presented. The approach is based on different node of Chebyshev polynomials, where the derivatives are expressed in terms of differentiation matrix, using the properties of Shifted Chebyshev polynomials of first kind, second kind and second derivative of first kind collocation nodes, together with Chebyshev collocation methods. This contribution offers comparison of the performance of different types of Chebyshev collocation nodes in comparison with exact solution, which shows that second derivative of first kind Chebyshev, is accurate and efficient, using two different error norms.
ultinomials, Multibinomial series.

Keywords:
Spectral Methods, Convection–diffusion equation, Chebyshev polynomials

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solving the linear system or eigenvalues problem that arises after discretization of the differential equation. Only iterative algorithms based on matrix-vector products, such as Conjugate gradients or GMRES for linear systems.

In this paper a numerical approximation of the solution $u:[0,1]\rightarrow \mathrm{R}$ of the following problem is consider:

$$\left.\begin{array}{l} -\varsigma u''(x)+\psi u'(x)=f(x), x\in[0,1] \\ u(0)=0 \\ u(1)=0 \end{array}\right\} \qquad (1)$$

The function *f* and the real numbers >0ς and ψ are given in such a way that there exists a unique continuous solution of this problem. Our aim is to approximate the solution with a continuous piecewise polynomial function. The differential equation in the problem (1) is an advection– diffusion equation. It models several phenomena, as, for example, the concentration of some chemical species transported in a fluid with speed ψ; the parameter ς is the diffusivity of the chemical species. The ratio =ψθς measures the importance of the advection compared to the diffusion. For large values of this ratio, the numerical solution of the problem (1) is delicate. The production and the vanishing of the chemical species are modeled by the function *f*, which in the general case depends on the unknown *u*. In this problem, we assume that *f* depends only on the position *x* and we consider ψ and ς as constants.

In order to solve (1) numerically, many researchers have used various numerical methods to solve the differential equation.Sonfyane and Bonlmalf (2005). Solution of linear and nonlinear parabolic equations by the decomposition method. Also Talbot and Crampton, applied pseudo-spectral method to 2D eigenvalues problem in elasticity using differentiation matrices. Talbot and Crampton (2005). Sapagovas (2002) introduced hypothesis on the solvability of parabolic equations with non local conditions.

$$\left.\begin{array}{l} x_j = cos\dfrac{(\pi * j)}{N} \quad j = 0,...,N \\ T_n(\lambda) = 2\lambda T_{n-1}(\lambda) - T_{n-2}(\lambda) \\ T_0(\lambda) = 1 \quad and \quad T_1(\lambda) = \lambda \end{array}\right\} \qquad (2)$$

Chebyshev polynomials of the second kind

The Chebyshev polynomials $U_n(x)$ of the second kind are orthogonal polynomials of degree n in x defined on the interval [–11] The polynomials $U_n(x)$ are orthogonal; with respect to the inner products

$$\left.\begin{array}{l} \int_{-1}^{1}\sqrt{1-x^2}U_n(x)U_m(x)dx = \begin{cases} 0, n \neq m, \\ \frac{\pi}{2}, n = m \end{cases} \\ U_n(x) = \dfrac{\sin(n+1)\theta}{\sin\theta} \end{array}\right\} \qquad (3)$$

The polynomials $U_n(x)$ are orthogonal on [-1, 1] with respect to the inner products

where $x = \cos(\theta)$ and $\theta \in [0,\pi]$. The polynomials $U_n(x)$ are orthogonal on $[-1,1]$ with respect to the inner products

$$U_n + 1(\lambda) = 2\lambda U_n(\lambda) - \lambda U_{n-1}(\lambda), \quad n = 1,2,... \qquad (4)$$

$$U_0(\lambda) = 1 \quad and \quad U_1(\lambda) = \lambda \qquad (5)$$

The nodes for the second Chebyshev polynomials is given as in [?]:

$$x_k = \cos\left(\frac{(k-1)\pi}{N-1}\right), \quad k = 1,..N \qquad (6)$$

Given a secod kind Chebyshev polynomials of degree N is denoted by $U_N(\lambda)$ on the int on the interval of $\{-1,1\}$.is defined in (4), the shifted second kind Chebyshev polynomials $U_N(x)$ can be genereted using the recurrence equation by introducing the $x = 2\lambda - 1$,Chebyshev polynomials are defined in the interval $(0,1)$ that may be called the second kind Chebyshev $U_N(x)$ that generate using the recurrence equation:

where the $\varphi_k(x)$ are Shifted second kind Chebyshev polynomials. A straightforward implementation of the spectrall methods involves the use spectral differentiation matrices to compute derivatives at the collocation pionts,in which $U=\{U(x_i)\}$ is the vector of values of $_N U$ at the $1N+$ collocation points and $U'=\{U(x_i)\}$ of the values of the derivatives at the collocation points, then the collocation derivatives matrix D is the matrix is mapping $U \to U'$ The entries of derivative matrix D can be obtained analytically. To obtain optimal accurracy this matrices must be computed carefully as suggested by Golbabai(2007). In Weideman(2000), the authors describe how to obtain it by the use of MATLAB code. We interpolate $u(x)$ by the shifted second kind Chebyshev polynomial $U_N(x)$ in the form equation (10). In the Chebyshev-Gauss-Lobatto points: where $\varphi_k(x)$ are the polynomial of degree $N-1$. The derivatives of the approximate solution $U_N(x)$ are then estimated at the colloction points by differentiation and evaluting, the resulting expression.This yields

$$u_N^n(x) \approx \sum_{k=0}^{N} a_k \varphi_k(x) \quad k = 0,\ldots,N$$

or in matrix notation

$$u^{(n)} = D^n U, n = 1, 2, \ldots\ldots$$

where

$U^{(n)} = [u_N^{(n)}(x_0), u_N^{(n)}(x_1), \ldots u_N^{(n)}(x_N)]^T, U = [u_N(x_0), u_N(x_0), \ldots, u_N(x_N)]^T$

where $D^{(n)}$ is the $(N+1)\times(N+1)$ matrix whose entries are given in this theorem, by Trefethen(2000).

Theorem 1 *For each N ...1, let the rows and columns of the $(N+1)\times(N+1)$ Chebyshev spectral differentiation matrix D_N be indexed from $0 to N$. The entries of this matrix are*

$$(D_N)_{00} = \frac{2N^2+1}{6}, \quad (D_N)_{NN} = -\frac{2N^2+1}{6} \tag{11}$$

$$(D_N)_{jj} = \frac{-x_j}{2(1-x_j^2)}, j = 1,\ldots,N-1 \tag{12}$$

$$(D_N)_{ij} = \frac{c_i(-1)^{i+j}}{c_j(x_i - x_j)}, i \neq j,\ i,j = 1,\ldots,N-1 \tag{13}$$

$$\begin{bmatrix} \varsigma D^{(2)}_{2:N-1} + \lambda D^{(2)}_{2:N-1} \\ I_{1,:} \\ I_{N,:} \end{bmatrix} \tag{22}$$

$$u = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

Zeros of different Chebyshev Collocation Nodes based on L_∞

The weighted monic polynomial are given in Mason(2005) as follows

$$T\_x(x) \Rightarrow x_k = \cos(\theta); \theta_k = \frac{(k)\pi}{n}, (k = 0, 1, \ldots n) \tag{24}$$

$$T'\_x(x) \Rightarrow x_t = -\cos(\theta); \theta_k = \frac{(k)\pi}{n}, (k = 0, 1, \ldots n) \tag{25}$$

$$U\_x(x) \Rightarrow x_r = \sin(\theta); \theta_r = \frac{(r\pi)}{n+2}, (r = 1, \ldots n+1) \tag{26}$$

In this paper, we consider the use of the first kind, second kind and Second derivative of first kind Collocation nodes, and exact solution to ascertain the efficiency of each of node, the efficiency is measure by the use of some error norms, which is state in section 5

Numerical Experiment

In this section, we present some numerical results to demonstrate the validity and effectiveness of different nodes and comparison with fist kind , derivative of first kind and second kind shifted Chebyschev polynomials. The accuracy efficiency are measured by the $L\infty$ error norm is measured:

$$L_\infty = \left\| u^{exact}(x_j) - u^{app}(x_j) \right\|_\infty = \max_{1, i, N} \left| u_i^{exact} - u_i^{app} \right| \tag{27}$$

$$L_\infty = \left\| \frac{u^{exact}(x_j) - u^{app}(x_j)}{u^{exact}(x_j)} \right\|_\infty = \max_{1, i, N} \left| \frac{u_i^{exact} - u_i^{app}}{u^{exact}(x_j)} \right| \tag{28}$$

where N is the number of interval points u_i^e and u_i are the exact and computed values of the solution u at point i.

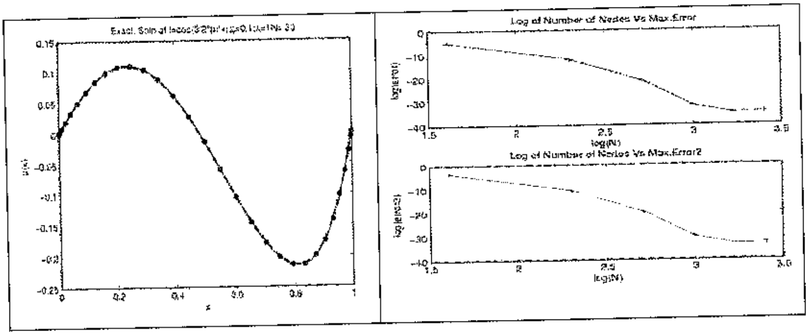

Figure 2: Num.Sol. vs Exac. Soln with node Error! Reference source not found. and Max.Error

Based on Derivative of second kind Chebyshevs Collocation points

Given the problem in (1) with $fc(x) = \cos(\frac{3}{2}\pi x); \lambda = 1; \varsigma = 0.1$

Table 3: Approximation error for u given in (1) using nodes of (26)

s/no	N	$\|u^n - u^e\|$	$\left\|\frac{\|u^n - u^e\|}{u^n}\right\|$
1	5	5.88e-3	3.26e-3
2	10	4.47e-6	2.00e-5
3	15	4.70e-10	2.22e-9
4	20	1.20e-14	5.80e-14
5	25	1.00e-15	3.00e-15

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